GEOPHYSICS, VOL. 72, NO. 2 (MARCH-APRIL 2007); P. A19–A23, 4 FIGS., 1 TABLE.
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# Fast finite-difference time-domain modeling for marine-subsurface electromagnetic problems

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## ABSTRACT

In the low-frequency limit, the displacement currents in the Maxwell equations can be neglected. However, for numerical simulations, a small displacement current should be present to achieve numerical stability. This requirement leads to a large range of propagation velocities with high velocities for the high frequencies and low velocities for the low frequencies. As a consequence, the number of time steps may become large. I show that it is possible to transform mathematically the original physical problem to one that has propagation velocities with less frequency dependence. Hence, the number of time steps necessary for a signal to travel a certain distance with the lowest velocity is significantly reduced. A typical example shows a reduction in computational time by a factor of 40. A comparison of the solutions from plane-layered modeling in the frequency and wavenumber domain and the proposed method shows good agreement between the two. The proposed method can also be used for other systems of diffusive equations.

## INTRODUCTION

Marine controlled-source electromagnetic surveying has been used for geophysical investigations since 1979 (Spies et al., 1980). A particular application, called seabed logging (SBL), was introduced by Eidesmo et al. (2002) and Ellingsrud et al. (2002) for offshore hydrocarbon exploration. The method consists of towing an electric dipole across the seafloor and recording the transmitted electromagnetic field at various positions. As hydrocarbon reservoirs are quite resistive compared to water-filled marine sediments (1–5 Ohm-m), very clear anomalies in magnitude versus offset are often seen. Both for feasibility studies and advanced processing of SBL data, a 3D modeling tool is of great importance. The migration proposed by Mittet et al. (2005), and also some inversion techniques, requires Green functions to be calculated at (in principle) each shot and receiver position. This task can be considerable, and methods for faster computations will increase the potential for such processing methods.

The finite-difference (FD) method for formulating and solving partial differential equations on computers has widespread applications. Yee (1966) introduced a time-domain FD (TDFD) method for solving Maxwell's equations. This method has since been applied successfully to many electromagnetic problems.

A TDFD method was introduced by Oristaglio and Hohmann (1984) for modeling the transient response in 2D conductive earth models. A main contribution was to introduce an artificially high electric permittivity. In effect, this approach increased the allowed (from the stability conditions) FD time-stepping and significantly reduced the overall computational time without changing the solution much. Another important contribution was to introduce the interaction with air by an upward continuation of the magnetic field (see Macnae, 1984). The 2D method by Oristaglio and Hohmann (1984) was later generalized to three dimensions by Wang and Hohmann (1993).

Berenger (1994) introduced the method of perfectly matched layers (PML), a method for implementing reflectionless boundaries in wave equations. The PML method provides a significant reduction of the size of the computational grid and has been generalized to conductive media (Chen et al., 1997; Liu, 1997).

An effective-medium theory for a general anisotropic, conductive layered earth was implemented by Davydycheva et al. (2003) on a Lebedev lattice (Lebedev, 1964). The Lebedev lattice defines all components of the electric field at given points in space, avoiding interpolation of the fields and their derivatives. An effective medium allows modeling the response of thin, resistive layers (e.g., hydrocarbon reservoirs) without reducing the spacing of the computational grid.

The upscaling of artificially high electric permittivity, the introduction of PML, and the introduction of effective media have all contributed to a reduction of computational time. Following the line of improvements for TDFD methods, I introduce a method for fur-

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ther reduction in the computational effort needed to solve Maxwell's equations in a geophysical setting. However, the proposed method can be used also to solve similar equations, like the diffusion equation.

An important obstacle for efficiently solving equations for geophysical problems with a TDFD method is the wide range of velocities involved in the problems. In the high-frequency limit, the propagation velocity is the speed of light. For low frequencies, propagation velocities depend upon frequency and conductivity of the earth. For the SBL method, low velocities are of the order $10^3$ m/s. Thus, the wide range of velocities dictates long simulation times with small time steps. This problem was partly resolved by introducing the quasistatic approximation in the air, and the artificially high electric permittivity (Oristaglio and Hohmann, 1984). These were approximations based on the underlying physics of the problem. In order to further reduce the discrepancy between high and low velocities (and reduce computational time), I propose to transform the original problem into another and solve Maxwell's equations in a medium that significantly differs from the original geophysical model. The solution for the modified problem then will be transferred back to give the solution of the original problem. In the original physical problem, I assume that the displacement current is negligible. Apart from that, the approach is not based on physical approximations, but purely on mathematical transformations.

## THEORY

### Mathematical transformation

I assume initially that Maxwell's equations for the physical problem can be written as

$$(\varepsilon_{ij}\partial_t + \sigma_{ij})E_j = \epsilon_{ijk}\partial_j H_k - J_i, \tag{1}$$

$$\mu\partial_t H_i = -\epsilon_{ijk}\partial_j E_k - M_i. \tag{2}$$

Here, $\varepsilon_{ij}$ is the electric permittivity tensor, and $\sigma_{ij}$ is the conductivity tensor. The $j$-components of the electric and magnetic fields are $E_j$ and $H_j$, respectively, and $\epsilon_{ijk}$ is the Levi-Civita tensor. The magnetic permeability $\mu$ is assumed constant and equal to $4\pi \times 10^{-7}$ H/m. The $i$-components of the electric and magnetic sources are denoted by $J_i$ and $M_i$, respectively. The Fourier transform of equations 1 and 2 from time to angular frequency yields

$$(-\iota\omega\varepsilon_{ij} + \sigma_{ij})E_j = \epsilon_{ijk}\partial_j H_k - J_i, \tag{3}$$

$$-\iota\omega\mu H_{\mathrm{i}} = -\epsilon_{ijk}\partial_j E_k - M_i. \tag{4}$$

I further assume that

$$|\iota\omega\varepsilon_{ij}E_j| \ll |\sigma_{ij}E_j| \tag{5}$$

for frequencies of interest. Thus, the solution to equations 3 and 4 can be found equally well from

$$-\iota\omega\mu\sigma_{ij}E_j = -\epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l E_m - M_k) + \iota\omega\mu J_i, \tag{6}$$

$$-\iota\omega\mu H_i = -\epsilon_{ijk}\partial_j(\boldsymbol{\sigma}^{-1})_{kl}(\epsilon_{lmn}\partial_m H_n - J_l) - M_i. \tag{7}$$

In TDFD, the internal time-stepping is restricted by the ratio of grid spacing and the fastest propagation velocity, known as the Courant-Friedriechs-Lewy stability condition for hyperbolic equations

$$\Delta t \leq \frac{1}{c\sqrt{\sum_{i=1,2,3}\frac{1}{(\Delta x_i)^2}}}, \tag{8}$$

where $\Delta t$ is the time-stepping, $\Delta x_i$ is the spatial grid size in direction $i$, and $c = 1/\sqrt{\mu\varepsilon}$ is the speed of light in the given medium. However, as long as equation 5 is fulfilled, the speed of light may be artificially reduced without changing the solution significantly and thus increase the internal time-stepping (according to equation 8). Consequently, computational time is reduced (see Oristaglio and Hohmann, 1984). In the following, I show how to go beyond the limitations given by equations 5 and 8 by solving a modified version of the physical problem. In the modified problem, I set $\varepsilon_{ij} = \alpha\sigma_{ij}$, where $\alpha$ is constant (in the physical problem, I assume that the effect of the electric permittivity tensor can be neglected, equation 5).Note that $\alpha$, in this way, determines the maximum propagation velocity in the modified problem. Equations 3 and 4 then become

$$(1 - \iota\omega'\alpha)\sigma_{ij}E'_j = \epsilon_{ijk}\partial_j H'_k - J'_i, \tag{9}$$

$$-\iota\omega'\mu H'_i = -\epsilon_{ijk}\partial_j E'_k - M'_i. \tag{10}$$

For clarity, I use primes in equations 9 and 10 to distinguish from quantities given in equations 3 and 4. Equations 9 and 10 can be combined to give

$$-\iota\omega'\mu(1 - \iota\omega'\alpha)\sigma_{ij}E'_j = -\epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l E'_m - M'_k) + \iota\omega'\mu J'_i, \tag{11}$$

$$-\iota\omega'\mu(1 - \iota\omega'\alpha)H'_i = -\epsilon_{ijk}\partial_j(\boldsymbol{\sigma}^{-1})_{kl}(\epsilon_{lmn}\partial_m H'_n - J'_l) - (1 - \iota\omega'\alpha)M'_i. \tag{12}$$

Comparing equation 11 with equation 6, I see that if

$$\iota\omega'(1 - \iota\omega'\alpha) = \iota\omega, \tag{13}$$

$$\iota\omega'\mu J'_i = \iota\omega\mu J_i, \tag{14}$$

$$M'_i = M_i, \tag{15}$$

then $E'_i = E_i$. Similarly, if

$$\iota\omega'(1 - \iota\omega'\alpha) = \iota\omega \tag{16}$$

$$J'_i = J_i, \tag{17}$$

$$(1 - \iota\omega'\alpha)M'_i = M_i, \tag{18}$$

then $H'_i = H_i$. Thus, by choosing appropriate $\alpha$, equations 9 and 10 can be solved in the time domain with a coarse time-stepping. In particular, $\alpha$ can be chosen such that $|\alpha\omega'| \gg 1$ for the frequencies of interest. This choice means that propagation velocities in the transformed problem have a weak frequency dependence. After simulation, the data can be Fourier transformed with a complex frequency according to equation 13. The desired electromagnetic fields (for the physical problem) are then found from the conditions given in equations 13-18.

### Finite-difference scheme

Equations 9 and 10 in the time domain read

$$(\alpha\partial_t + 1)\sigma_{ij}E_j = \epsilon_{ijk}\partial_j H_k - J_i \tag{19}$$

$$\mu\partial_t H_i = -\epsilon_{ijk}\partial_j E_k - M_i. \tag{20}$$

The discretization of equations 19 and 20 in time and space is quite straightforward. I use the same grid definition as in Yee (1966), and the spatial derivatives are approximated by differences on a regular grid. However, it is worth noting the difference between equations 19 and 1 in their numerical implementation. Equation 1 is discretized in time to give

$$E_i^{n+1} = A_{ij}E_j^n + B_{ij}(\epsilon_{jkl}\partial_k H_l^{n+1/2} + J_j^{n+1/2}), \qquad (21)$$

where

$$B_{ij} = (\boldsymbol{\varepsilon}/\Delta t + \boldsymbol{\sigma}/2)_{ij}^{-1}, \qquad (22)$$

$$A_{ij} = B_{ik}(\boldsymbol{\varepsilon}/\Delta t - \boldsymbol{\sigma}/2)_{kj}. \qquad (23)$$

Because of the low-frequency assumption in equation 5, the tensor $\boldsymbol{\varepsilon}$ can be replaced by a scalar, $\varepsilon$. In order to evaluate equation 21, interpolation of $E_{j\neq i}^n$ must be performed for each time step (Weiss and Newman, 2002). For the proposed method, equation 19 is discretized in time to give

$$E_i^{n+1} = \frac{2\alpha - \Delta t}{2\alpha + \Delta t}E_i^n + B_{ij}(\boldsymbol{\varepsilon}_{jkl}\partial_k H_l^{n+1/2} + J_j^{n+1/2}), \quad (24)$$

where

$$B_{ij} = \frac{2\Delta t(\boldsymbol{\sigma}^{-1})_{ij}}{2\alpha + \Delta t}. \qquad (25)$$

Comparing equation 24 with 21, the new scheme requires less memory and less interpolation. The only interpolations needed are those of the terms involving $\partial_k H_l$.

## NUMERICAL EXAMPLES

A 3D version of the method just described was implemented, tested, and compared with the standard 3D TDFD, as well as plane-layer modeling in the frequency and wavenumber domain. Modeling was performed with a plane-layered model; a regular grid with 100-m spacing in all three dimensions was used. The model is shown in Figure 1. The horizontal electric-dipole source is placed 30 m above seafloor level, pointing along the source-receiver line. The electromagnetic field is recorded at the seafloor at stations with 100-m horizontal intervals along a line. For FD modeling, I used second-order spatial operators (for positioning and differencing).

Figure 2a shows the resulting inline transient response of the electric field, at 10-km offset from a standard TDFD simulation. By a standard TDFD, I mean the use of a constant artificially high electric permittivity (Oristaglio and Hohmann, 1984; Wang and Hohmann, 1993). The modeling must be performed to large enough times for the transients to decay sufficiently. Otherwise, noise may be introduced in the Fourier transform when data in the frequency domain are examined.

The corresponding response from the proposed method is shown in Figure 2b. Clearly, the results are very different. In contrast to the standard TDFD simulation, it is unnecessary to wait for the propagating signal to decay. The complex frequency used in the Fourier transform will suppress the later arrivals with exponential decay. The number of time steps used by the proposed method was approximately 40 times fewer than that used by the standard TDFD. Temporal discretization, the number of time steps, and the computational times are given in Table 1. The size of the model is $N_x = 211$, $N_y = 15$, and $N_z = 41$, including the absorbing boundaries (PML) of five grid points at each side. Size in the transverse direction was strongly reduced because of the simplicity of the problem (2.5D). A true 3D model ($N_y = 211$) would give an increase in the computational time by a factor 14 for both TDFD methods.

The data were transformed to the frequency domain, and equations 13-18 were applied on data from the proposed method. The transformed data were compared with those from plane-layer modeling in the frequency and wavenumber domain (*f-k* modeling) and with standard 3D TDFD data. Figure 3 shows the amplitudes and phases at 0.25 Hz. No significant differences between the different methods are found, except at the very near offsets. This similarity is a result of the spatial discretization and reflects the difference between a point source (*f-k*) and a band-limited point source (FD). In real experiments, the source dipole is usually longer than 200 m.

Figure 4 displays the amplitudes and phases at 2.5 Hz. At this frequency, the standard 3D TDFD modeling begins to deviate from the other two at large offsets. This effect is most clearly seen as a slight phase error. The accuracy of the modeled magnetic field is similar to that of the electric field.

## DISCUSSION

The main difference between the time-domain results of the standard method and of the one proposed is given by the nature of the processes they describe. The original physical problem is governed by a high degree of attenuation, both in time and space. The modified equations describe a system with less damping, where multiple reflections can propagate easily through the medium, similar to what we have in seismic propagation. Thus, it is not intuitive that, with the proper transformation, the two methods should give very similar results.

For the method described by Oristaglio and Hohmann (1984) and Wang and Hohmann (1993), I have used a relative permittivity of $9.6 \times 10^6$, which corresponds to a maximum time step of $3.1 \times 10^{-4}$ s. This time step is 10 times smaller than what is recommended as sufficient in Wang and Hohmann (1993). Even so, I can see a deviation from the *f-k* modeling at larger offsets for the frequency of 2.5 Hz. Increasing the simulated time from 50 to 100 s does not change the result and does not cause the deviation. A further reduc-

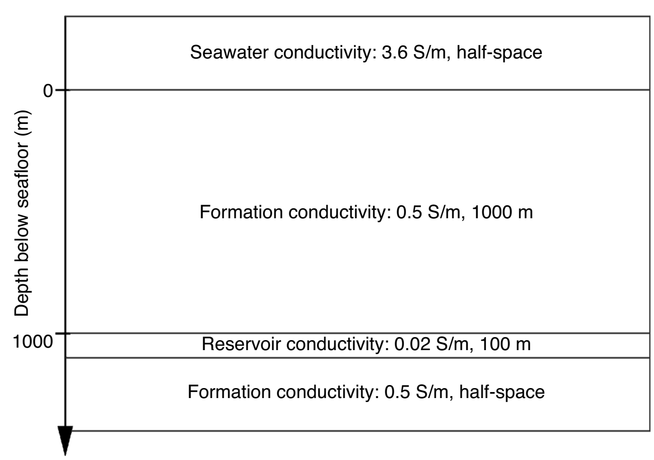


Figure 1. The plane-layered model used for comparison of results. The model consists of a semi-infinite water layer, a 1-km formation layer, a 100-m resistive layer, and a semi-infinite formation layer.

tion in simulation time will cause reduction in accuracy. However, it is possible to reduce the discrepancy by reducing the artificially high permittivity, although this approach results in increased computational time. Thus, the deviation is mainly caused by the artificial displacement current. This conclusion does not imply that the guidelines given in Wang and Hohmann (1993) are wrong, because the amplitudes at 10-km offset are three orders of magnitude smaller at 2.5 Hz than those at 0.25 Hz. Thus, the difference in the time-domain data will be very small.

The transformations given in equations 13-18 yield the solution to equations 3 and 4, i.e., a solution in which displacement currents are negligible. For the modified DuFort-Frankel method (Oristaglio and Hohmann, 1984; Wang and Hohmann, 1993), $\varepsilon$ must be chosen such that equation 5 is satisfied. Thus, the modified DuFort-Frankel method requires careful treatment of the higher frequencies. This problem is omitted with the proposed method. For $\alpha\omega \ll 1$, the proposed method is equivalent to the modified DuFort-Frankel method (although the proposed method does not need to perform interpolation of the electric field to handle anisotropy) and gives similar time-domain solutions.

Introduction of $\alpha$ reduces the highest-propagation speed in the model, which can be seen as a late first arrival in Figure 2b. It also introduces less attenuation of the propagated signal. As can be seen from Figure 2b, the numerical experiment is far from relaxed. However, this is not necessary because the complex frequency used in the Fourier transform attenuates the latest arrivals, and the contribution to the Fourier transform is eventually negligible. It is therefore clear that the required simulation time depends on the frequency content we want to study. It is instructive to note that the propagation velocities scale as

$$c \approx 1/\sqrt{\alpha}. \tag{26}$$

Thus, the minimum time $T_0$ necessary for the signal to reach a given destination will scale as

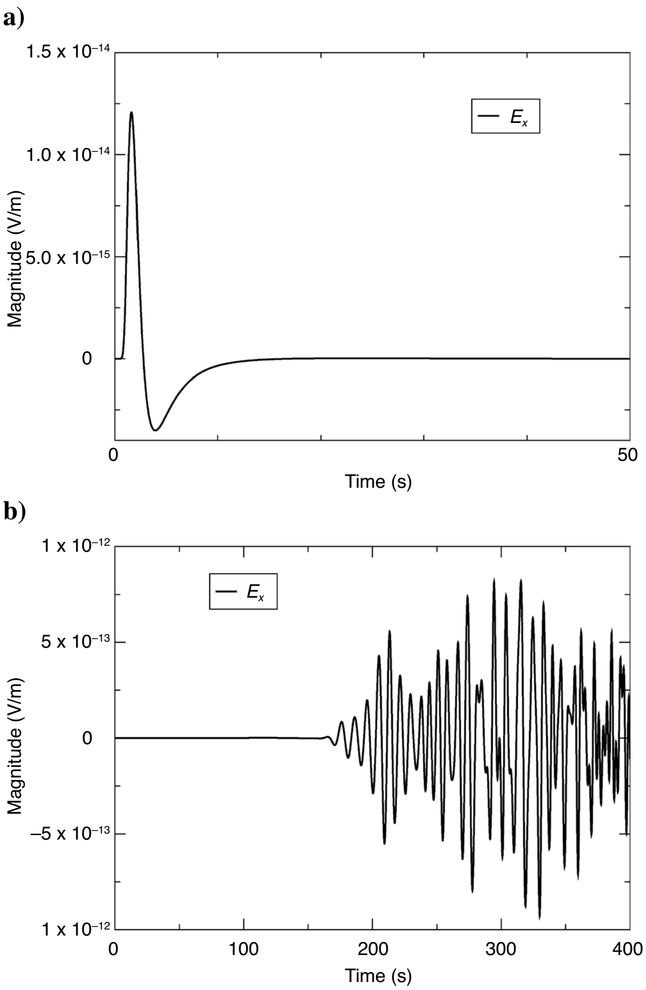


Figure 2. Resulting time series for the inline electrical component $E_x$ at 10-km offset using (a) the standard method and (b) the proposed method.

**Table 1. Time discretization $\Delta t$, number of iterations used $N_t$, and simulation times $T_{\text{cpu}}$ for the given FD examples. Simulation times are for a single 3.4-GHz CPU.**

| | $\Delta t$ (s) | $N_t$ | $T_{\text{cpu}}$ (min) |
|---|---|---|---|
| Old | $3.1 \times 10^{-4}$ | $1.7 \times 10^5$ | 121 |
| New | 0.1 | 4001 | 3.0 |

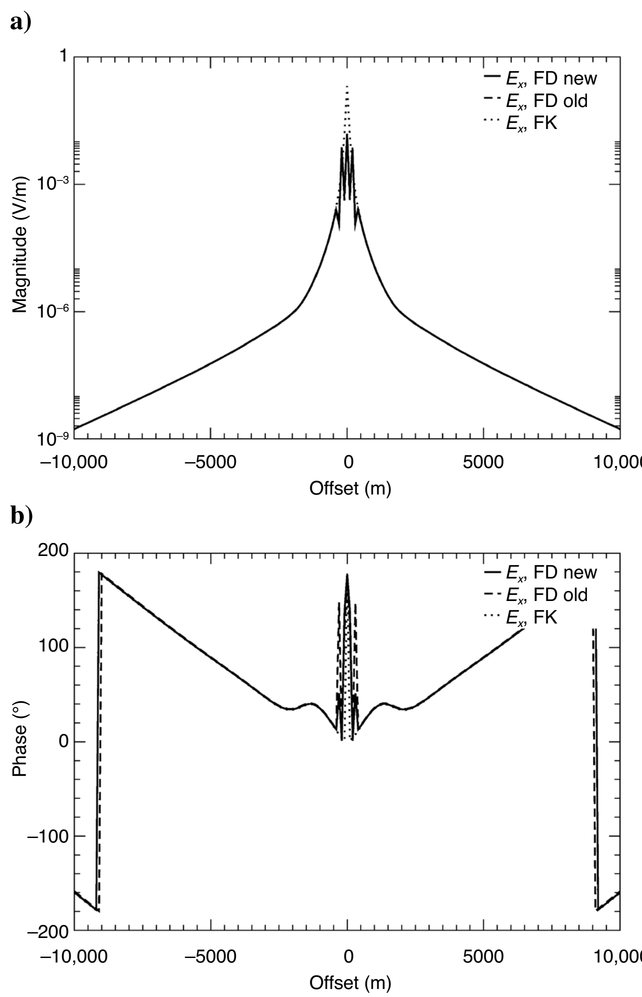


Figure 3. (a) Amplitudes and (b) phases in the frequency domain (0.25 Hz) for the proposed method, standard TDFD, and plane-layer modeling FK.

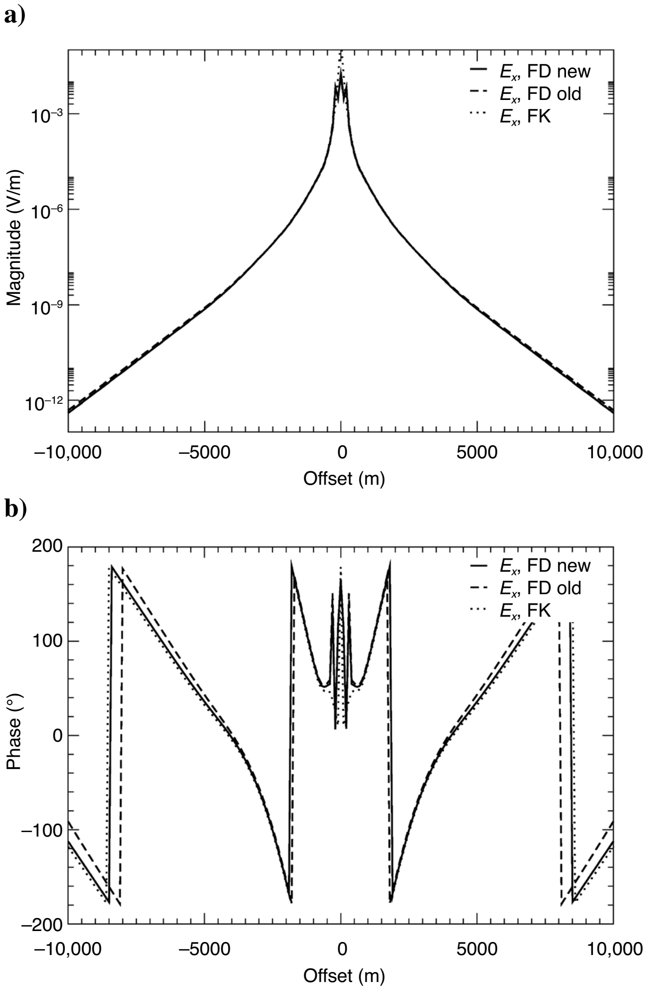


Figure 4. (a) Amplitudes and (b) phases in the frequency domain (2.5 Hz) for the proposed method, standard TDFD, and plane-layer modeling FK.

$$T_0 \approx \sqrt{\alpha}. \tag{27}$$

Note that $T_0$ is not the necessary time to have a convergent result. The CFL condition, equation 8, implies that the time-stepping scales as

$$\Delta t \approx \sqrt{\alpha}. \tag{28}$$

Thus, the minimum number of iterations required, $T_0/\Delta t$, will be a constant. The key point is to reduce numerical dispersion, such that signals of different frequencies arrive at similar times.

In general, the internal time-stepping $\Delta t$ is dictated by the fastest propagation speed, whereas the total simulation time is set by the lowest propagation speed (or event). The proposed method will reduce the gap between these limiting factors but will not remove it. In particular, the frequency dependence of velocities will be small when $|\alpha\omega'| \gg 1$ for the frequencies of interest. Speedup will therefore vary from case to case and depend upon the chosen value of $\alpha$, the desired frequency range, and the complexity of the model. The reduction in computational time was 40 times for the example given.

## CONCLUSIONS

In a situation where the displacement currents are negligible, I have shown that it is possible to transform the original Maxwell's equations of the physical problem into a set of equations in which the displacement currents can be substantial. Hence, the frequency dependence of the propagation velocities can be reduced significantly. The finite-difference time step is given by the largest velocity, and the total simulation time is given by the smallest velocity. Thus, a significant decrease in computational time is achieved. This result can have great impact on the applicability of 3D modeling tools, such as survey planning, imaging, and inversion.

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